Birkhoff and New Orthogonality in Normed Linear Spaces Via 2-HH Norm

Bhuwan Prasad Ojha , Prakash Muni Bajracharya

Abstract

The p-HH norms were introduced by Kikianty and Dragomir on the Cartesian square of normed spaces . P-norms and p-HH norms induces the same topology, so they are equivalent, but geometrically they are different. Besides that, E. Kikianty and S.S. Dragimor introduced HH-P orthogonality and HH-I orthogonality by using 2-HH norm and discussed main properties of these orthogonalities. In this paper, we test the concept of 2-HH norm to Birkhoff and a new orthogonality in normed spaces and discuss some properties of these orthogonalities.

Keywords: *Birkhoff orthogonality,Hermite-Hadamard's inequality, Pythagorean orthogonality, p-HH norm, Logarithmic mean*

1 Introduction

An inner-product on X defines a norm on X by $\|x\|^2 = \langle x, x\rangle$. Every innerproduct spaces are normed spaces, but the converse may not be true. A best example of normed space which is not an inner-product space is $l^p = \{(x_n), x_n \in \mathbb{R} : \sum |x_n| < \infty\}$ for $p \neq 2$.

Definition. The $p - HH$ norm on $X^2 = X \times X$ is defined by

$$\|(x, y)\|_{p-HH} = (\int_0^1 \|(1-t)x + ty\|^p \, dt)^{\frac{1}{p}}$$

for any $x, y \in X^2$ and $1 \leq p < \infty$.

The 2-HH norm is defined as follows:

$$\begin{aligned} \|(x, y)\|_{2-HH}^2 &= \int_0^1 \|(1-t)x + ty\|^2 \, dt \\ &= \frac{1}{3}[\|x\|^2 + \langle x, y\rangle + \|y\|^2 \end{aligned}$$

The p-HH norms are equivalent to p-norms on X^2, as they induce the same topology, but geometrically they are different. The p-HH norm is an extension of the generalized logarithmic mean which is connected by the Hermite-Hadamards inequality to p-norm. The definition of the generalized logarithmic mean and Hermite-Hadamards inequality are as follows:

Definition. [12] For any convex function $f : [a, b] \to \mathbb{R}([a, b] \subset \mathbb{R}$, the Hermite-Hadamard's inequality is defined as

$$(b-a)f(\frac{a+b}{2}) \leq \int_a^b f(t)dt \leq (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

. This inequality has been extended (see-12) for convex function $f : [x, y] \to \mathbb{R}$, where $[x, y] = \{(1-t)x + ty, t \in [0, 1]\}$. In that case Hermite-Hadamards integral inequality becomes

$$f(\frac{x+y}{2)} \leq \int_0^1 f\left[(1-t)x+ty\right]dt \leq \frac{f(x)+f(y)}{2} \qquad ......(1).$$

Using the convexity of $f(x) = \|x\|^p \quad (x \in X, p \geq 1)$ and relation (1) we have

$$\left\|\frac{x+y}{2}\right\| \leq \left[\int_0^1 \|(1-t)x+ty\|^p\, dt\right]^{\frac{1}{p}} \leq \frac{1}{2^{\frac{1}{p}}}(\|x\|^p + \|y\|^p)^{\frac{1}{p}}.$$

1.1 HH-P Orthogonality

Definition. [3, 4] A vector x is said to be orthogonal to y in the sense of Pythagorean if $\|x-y\|^2 = \|x\|^2 + \|y\|^2$.

[8] Let $(X, \|.\|)$ be a normed space. Then $x \perp_{HH-P} y \Longleftrightarrow \int_0^1 \|(1-t)x+ty\|^2\, dt = \frac{1}{3}(\|x\|^2 + \|y\|^2)$.

1.1.1 Properties of HH-P orthogonality

1. HH-P orthogonality satisfies non-degeneracy, simplification, continuity and symmetry.
2. HH-P orthogonality is existent.
3. HH-P orthogonality is unique.
4. HH-P orthogonality is homogeneous if and only if the space is inner-product space.
5. HH-P orthogonality is additive if the space is an inner-product space.

1.2 HH-I orthogonality

Definition. [5] A vector x is said to be isosceles orthogonal to y if $\|x-y\| = \|x+y\|$.

[8] Let $x, y \in X$ such that $\|(1-t)x+ty\| = \|(1-t)x-ty\|$ a.e. on $[0, 1]$. Then x is said to be HH-I orthogonal to y iff

$$\int_0^1 \|(1-t)x+ty\|\, dt = \int_0^1 \|(1-t)x-ty\|\, dt.$$

1.2.1 Properties of HH-I Orthogonality

1. The HH-I orthogonality satisfies non-degeneracy, simplification, continuity and symmetry properties.
2. HH-I orthogonality is existent.
3. If HH-I orthogonality is homogeneous in a normed space X, then X is an inner-product space.
4. If HH-I orthogonality is additive, then the space is an inner-product space.
5. HH-I orthogonality is neither right additive nor homogeneous.

Definition. [2] In a normed linear space X,

$$x \perp y \Leftrightarrow \sum_{k=1}^{m} a_k \|b_k x + c_k y\|^2 = 0,$$

where $m \geq 2$ and a_k, b_k, c_k are real numbers such that

$$\sum_{k=1}^{m} a_k b_k c_k \neq 0, \quad \sum_{k=1}^{m} a_k b_k^2 = \sum_{k=1}^{m} a_k c_k^2 = 0$$

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1.3 HH-C Orthogonality

[8] Let $(X, \|.\|)$ be a normed space and $t \in [0,1]$. then $x \in X$ is said to be HH-C orthogonal to to $y \in X$ if and only if

$$\sum_{j=1}^{m} \alpha_j \int_0^1 \|(1-t)\beta_j x + t\gamma_j y\|^2 = 0$$

satisfying the conditions

$$\sum_{j=1}^{m} \alpha_j \beta_j \gamma_j \neq 0 \quad \text{and} \quad \sum_{j=1}^{m} \alpha_j \beta J_{\mathrm{J}}{}^2 = \sum_{j=1}^{m} \alpha_j \gamma_j^2 = 0.$$

HH-P orthogonality is a particular case of HH-C orthogonality

Let us take

$$\sum_{j=1}^{3} \alpha_j \int_0^1 \|(1-t)\beta_j x + t\gamma_j y\|^2 = 0$$

$$\Rightarrow \alpha_1 \int_0^1 \|(1-t)\beta_1 x + t\gamma_1 y\|^2 \, dt + \alpha_2 \int_0^2 \|(1-t)\beta_2 x + t\gamma_2 y\|^2 \, dt + \alpha_3 \int_0^1 \|(1-t)\beta_3 x + t\gamma_3 y\|^2 \, dt = 0$$

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Taking $\alpha_1 = -1$, $\alpha_2 = \alpha_3 = 1$, $\beta_1 = \beta_2 = 1$, $\beta_3 = 0$, $\gamma_1 = \gamma_3 = 1$ and $\gamma_2 = 0$, we get

$$-\int_0^1 \|(1-t)x + ty\|^2 \, dt + \int_0^1 \|(1-t)x\|^2 \, dt + \int_0^1 \|ty\|^2 \, dt = 0$$

$$\Rightarrow -\int_0^1 \|(1-t)x + ty\|^2 \, dt + \frac{1}{3}(\|x\|^2 + \|y\|^2 = 0$$

$$\therefore \quad \int_0^1 \|(1-t)x + ty\|^2 \, dt = \frac{1}{3}(\|x\|^2 + \|y\|^2$$

Now

$$\sum_{k=1}^{3} \alpha_j \beta_j \gamma_j = \alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2 + \alpha_3\beta_3\gamma_3 = -1, \quad \sum_{j=1}^{m} \alpha_j \beta j^2 \quad = \alpha_1\beta 1^2 + \alpha_2 \beta 2^2 + \alpha_3 \beta 3^2 = 0$$

$$\text{and} \quad \sum_{j=1}^{m} \alpha_j \gamma_j^2 = \alpha_1\gamma_1^2 + \alpha_2\gamma_2^2 + \alpha_3\gamma_3^2 = 0$$

Which shows that HH-P orthogonality is a particular case of HH-C orthogonality.

HH-I orthogonality is a particular case of HH-C orthogonality

Let us take

$$\sum_{j=1}^{2} \alpha_j \int_0^1 \|(1-t)\beta_j x + t\gamma_j y\|^2 = 0$$

$$\Rightarrow \alpha_1 \int_0^1 \|(1-t)\beta_1 x + t\gamma_1 y\|^2 \, dt + \alpha_2 \int_0 \|(1-t)\beta_2 x + t\gamma_2 y\|^2 \, dt = 0$$

Taking $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{-1}{2}, \beta_1 = \beta_2 = 1, \gamma_1 = 1, \gamma_2 = -1$, we get

$$\frac{1}{2}\int_0^1 \|(1-t)x + ty\|^2 \, dt - \frac{1}{2}\int_0^1 \|(1-t)x - ty\|^2 \, dt = 0$$

$$\Rightarrow \int_0^1 \|(1-t)x + ty\|^2 \, dt = \int_0^1 \|(1-t)x - ty\|^2 \, dt$$

Now

$$\sum_{k=1}^{2} \alpha_j \beta_j \gamma_j = \alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2 = 1, \quad \sum_{k=1}^{2} \alpha_j \beta_j^2 = \alpha_1\beta_1^2 + \alpha_2\beta_2^2 = 0$$

$$\text{and} \quad \sum_{k=1}^{2} \alpha_j \gamma_j^2 = \alpha_1\gamma_1^2 + \alpha_2\gamma_2^2 = 0$$

1.3.1 Properties of HH-C orthogonality

1. HH-C orthogonality satisfies non-degeneracy, simplification , and continuity property.
2. HH-C orthogonality is not symmetric.
3. HH-C orthogonality is neither additive nor homogeneous.

2 Main Result

Definition. [11] A vector x is orthogonal to y if

$$\left\|x+\frac{1}{2}y\right\|^2+\left\|x-\frac{1}{2}y\right\|^2=\frac{1}{2}\left\|\sqrt{2}x+y\right\|^2+\|x\|^2$$

Lemma 2.1. *For an abstract Euclidean Space X, orthogonality relation* $\left\|x+\frac{1}{2}y\right\|^2+\left\|x-\frac{1}{2}y\right\|^2=\frac{1}{2}\left\|\sqrt{2}x+y\right\|^2+\|x\|^2$ *implies Birkhoff orthogonality if* $y=\frac{x}{1-\alpha}$.

Proof. Suppose $x\perp y$. Then by definition,

$$\left\|x+\frac{1}{2}y\right\|^2+\left\|x-\frac{1}{2}y\right\|^2=\frac{1}{2}\left\|\sqrt{2}x+y\right\|^2+\|x\|^2$$
$$\Rightarrow\left\|x+\frac{1}{2}y\right\|^2+\left\|x-\frac{1}{2}y\right\|^2\geq\|x\|^2$$
$$\Rightarrow\left\|x+\frac{1}{2}y-x+\frac{1}{2}y\right\|^2\geq\|x\|^2$$
$$\Rightarrow\|y\|^2\geq\|x\|^2 \ldots\ldots. \qquad (1)$$

Since $y=\frac{x}{1-\alpha}$ so that $y=x+\alpha y$. Therefore form the relation (1)

$$\|x+\alpha y\|^2\geq\|x\|^2$$
$$\Rightarrow\|x+\alpha y\|\geq\|x\|$$
$$\Rightarrow x\perp_B y.$$

□

But the converse of above lemma may not be true. Consider $X=(\mathbb{R}^2,\|.\|_1)$, where $\|.\|_1=\sum_{k=1}^{2}|x_k|$ for some $x=(x_1,x_2)\in X$. Let $x=(-2,1), y=(2,2)$. and $\alpha\in\mathbb{R}$ we have

$$\|x+\alpha y\|_1=\|(2,1)+\alpha(2,2)\|_1 \quad =\|-2+2\alpha,1+2\alpha\|_1=|-2+2\alpha|+|1+2\alpha| \quad \geq 3=\|x\|_1$$

But

$$\begin{aligned}\left\|x+\frac{1}{2}y\right\|^2+\left\|x-\frac{1}{2}y\right\|^2&=\left\|(-2,1)+\frac{1}{2}(2,2)\right\|^2+\left\|(-2,1)-\frac{1}{2}(2,2)\right\|^2\\&=\|(-2,1)+(1,1)\|^2+\|(-2,1)-(1,1)\|^2\\&=18\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\left\|\sqrt{2}x+y\right\|^2+\|x\|^2&=\frac{1}{2}\left\|\sqrt{2}(-2,1)+(2,2)\right\|^2+\|(-2,1)\|^2\\&=\frac{1}{2}\left\|(-2\sqrt{2}+2,\sqrt{2}+2)\right\|^2+9\\&=\frac{1}{2}(0.828+3.4142)^2+9\\&=17.99\end{aligned}$$

which shows that x is not orthogonal to y in the sense of above orthogonality.

3 Birkhoff Orthogonality Via 2-HH norm

Definition. [6, 9] A vector x is said to be orthogonal to y in the sense of Birkhoff if $\|x\| \leq \|x+\alpha y\|$ for all $\alpha \in \mathbb{R}$.

In the case of $2-HH$ norm,

$$\begin{aligned}\int_0^1 \|(1-t)x+\lambda ty\|^2 &= \int_0^1 \langle (1-t)x+\lambda ty, (1-t)x+\lambda ty\rangle dt \\ &= \|x\|^2 \int_0^1 (1-t)^2 dt + 2\lambda\langle x, y\rangle \int_0^t t(1-t)dt + \lambda^2 \|y\|^2 \int_0^1 t^2 dt.\end{aligned}$$

If $x\perp$, then

$$\begin{aligned}\int_0^1 \|(1-t)x+\lambda ty\|^2 &= \|x\|^2 \int_0^1 (1-t)^2 dt + \lambda^2 \|y\|^2 \int_0^1 t^2 dt \\ &= \frac{1}{3}(\|x\|^2 + \|\lambda y\|^2) \qquad ... \quad (1)\end{aligned}$$

But $\int_0^1 \|(1-t)x\|^2 dt = \|x\|^2 \int_0^1 (1-t)^2 dt = \frac{1}{3}\|x\|^2.$ ... (2)
Since $\|\lambda y\|^2$ is a non-negative quantity, so from relation (1) and (2), we conclude that

$$\int_0^1 \|(1-t)x+\lambda ty\|^2 \geq \int_0^1 \|(1-t)x\|^2 dt. \qquad ... \quad (3)$$

Keeping the above result in our mind, we can conclude that $x\perp_2 - HH(B)y$ if the relation (3) is satisfied.

4 New Orthogonality Via 2-HH Norm

[11] A vector $x \in X$ is said to be orthogonal to the vector $y \in Y$ if and only if

$$\left\|x+\frac{1}{2}y\right\|^2 + \left\|x-\frac{1}{2}y\right\|^2 = \frac{1}{2}\left\|\sqrt{2}x+y\right\|^2 + \|x\|^2.$$

Using the concept of $2-HH$ norm,

$$\left\|x+\frac{1}{2}y\right\|^2 + \left\|x-\frac{1}{2}y\right\|^2 = \frac{1}{2}\left\|\sqrt{2}x+y\right\|^2 + \|x\|^2 \text{ a.e on } \quad [0,1]$$

and we obtain a definition of new orthogonality by using 2-HH norm is as follows: $x\perp y$ iff

$$\int_0^1 \left\|(1-t)x+\frac{1}{2}ty\right\|^2 dt + \int_0^1 \left\|(1-t)x-\frac{1}{2}ty\right\|^2 dt = \frac{1}{2}\int_0^1 \left\|\sqrt{2}(1-t)x+ty\right\|^2 dt + \int_0^1 \|(1-t)x\|^2 dt.$$

.........(1)

To verify the above definition, the left hand side of relation (1)

$$\begin{aligned}\int_0^1 \left\|(1-t)x+\frac{1}{2}ty\right\|^2 dt+\int_0^1 \left\|(1-t)x-\frac{1}{2}ty\right\|^2 dt &= \int_0^1 \langle(1-t)x+\frac{1}{2}ty,(1-t)x+\frac{1}{2}ty\rangle dt\\ &+\int_0^1 \langle(1-t)x-\frac{1}{2}ty,(1-t)x-\frac{1}{2}ty\rangle dt\\ &=\frac{1}{3}\|x\|^2+\frac{1}{12}\|y\|^2+\frac{1}{3}\|x\|^2+\frac{1}{12}\|y\|^2\\ &=\frac{2}{3}\|x\|^2+\frac{1}{6}\|y\|^2.\end{aligned}$$

Again the right hand side of relation (1)

$$\begin{aligned}\frac{1}{2}\int_0^1 \left\|\sqrt{2}(1-t)x+ty\right\|^2 dt+\int_0^1 \|(1-t)x\|^2 dt &= \frac{1}{2}\int_0^1 \langle\sqrt{2}(1-t)x+ty,\sqrt{2}(1-t)x+ty\rangle dt+\frac{1}{3}\|x\|^2\\ &=\frac{1}{2}(\frac{2}{3}\|x\|^2+\frac{1}{3}\|y\|^2)+\frac{1}{3}\|x\|^2\\ &=\frac{2}{3}\|x\|^2+\frac{1}{6}\|y\|^2.\end{aligned}$$

Data Availability

There is not use of any data for the completion of this study.

Conflict of Interest

We authors do no have a conflict of interest for the publication of article.

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